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Generalised KdV and MKdV equations associated with symmetric spaces

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Abstract. We extend previous results on the linear spectral problem introduced by Fordy and Kulish. The odd-order isospectral flows admit both a KdV and MKdV type reduction. The non-linear terms are related to the curvature tensor of the corresponding Hermitian symmetric space. Our KdV equations are themselves reductions of known matrix KdV equations. We discuss the conserved densities and Hamiltonian structure associated with these equations.

1. Introduction

It is well known (Wadati and Kamijo 1974, Calogero and Degasperis 1977) that the third-order isospectral flow of the matrix Schrödinger equation:

$$(\partial^2 - U)\psi = \lambda^2\psi \tag{1.1}$$

with U an $N \times N$ real matrix, is the matrix KdV equation

$$U_{t_3} = U_{xxx} - 3(UU_x + U_xU). \tag{1.2}$$

In general this is a system of N^2 coupled (scalar) equations, but often only the symmetric case is considered, reducing the number of equations to $\frac{1}{2}N(N+1)$.

In this paper we associate a matrix KdV equation with each of the Hermitian symmetric spaces, thus enabling the matrix U to have as few as N independent components, giving rise to a significantly reduced system (1.2). We similarly associate a generalised MKdV equation with these symmetric spaces which, for class CI spaces, is related to (1.2) through a matrix Miura transformation. Both classes of equation are reductions of a general third-order isospectral flow of the linear spectral problem presented in Fordy and Kulish (1983) in the context of generalised NLS equations. In the case of matrix KdV equations this spectral problem can be rewritten as a matrix Schrödinger equation as in (1.1). In the symmetric space coordinates the non-linear parts of the equations are written in terms of the corresponding curvature tensor, as in (2.6).

Our general third-order system is Hamiltonian and possesses an infinite number of conserved densities. In the KdV reduction half of these densities remain non-trivial, corresponding to the known (Olmedilla *et al* 1981) densities for the matrix Schrödinger equation (1.1). However, in terms of the symmetric space coordinates our generalised KdV equations are not usually Hamiltonian, even though they can be squeezed between two matrix KdV equations, each of which *is* Hamiltonian. In the MKdV reduction half of the above-mentioned conserved densities also remain non-trivial, but in this case there appears to be no Hamiltonian structure at all.

2. Symmetric spaces and the spectral problem

First we review some of the basic facts concerning Hermitian symmetric spaces. More details can be found in Fordy and Kulish (1983) and Helgason (1978). We shall not be concerned with the geometry of these symmetric spaces but only with the associated splitting of the corresponding (semi-)simple Lie algebra

$$\mathbf{g}=\mathbf{k}\oplus\mathbf{m}$$

where $\mathbf{k}$ is a subalgebra of $\mathbf{g}$ and $\mathbf{m}$ (corresponding to the tangent space at point p of the symmetric space), the complementary subspace of $\mathbf{k}$ in $\mathbf{g}$. The full relationships satisfied by $\mathbf{k}$ and $\mathbf{m}$ are

$$[\mathbf{k},\mathbf{k}]\subset\mathbf{k}\qquad[\mathbf{k},\mathbf{m}]\subset\mathbf{m}\qquad[\mathbf{m},\mathbf{m}]\subset\mathbf{k}.$$

A special feature of Hermitian symmetric spaces is the existence of an element $A\in\mathbf{k}$ such that $\mathbf{k}=C_{\mathbf{g}}(A)=\{B\in\mathbf{g}\colon[A,B]=0\}$. The element A can (and therefore will) be chosen to be diagonal: $A\in\mathbf{h}\subset\mathbf{k}\subset\mathbf{g}$ where $\mathbf{h}$ is the Cartan subalgebra of $\mathbf{g}$. This element is highly degenerate; indeed, adA (which is a $\dim\mathbf{g}\times\dim\mathbf{g}$ matrix) has only three distinct eigenvalues: $0, \pm a$. Specifically, we have $\mathbf{m}=\mathbf{m}^+\oplus\mathbf{m}^-$ and $[A,\mathbf{k}]=0, [A,X^\pm]=\pm aX^\pm$ with a being the same value for all $X^\pm\in\mathbf{m}^\pm$.

With A defined as above and $Q(x,t)\in\mathbf{m}$ consider the linear spectral problem:

$$\psi_x=(\lambda A+Q)\psi \tag{2.1}$$

where λ is the spectral parameter and t the time parameter defined by the linear evolution equation:

$$\psi_t=P(x,t;\lambda)\psi. \tag{2.2}$$

The integrability conditions of (2.1) and (2.2), together with the isospectral condition $\lambda_t=0$, imply

$$Q_t=P_x-[\lambda A+Q,P] \tag{2.3}$$

which, for certain choices of P, is a system of non-linear evolution equations, exactly soluble by means of an inverse spectral transform. With $Q=Q^++Q^-\equiv\Sigma_\alpha(q^\alpha e_\alpha+r^{-\alpha}e_{-\alpha})$, where $\{e_{\pm\alpha}\}$ are the basis for $\mathbf{m}^\pm$ and $P=P^0+P^++P^-$, equation (2.3) becomes

$$\begin{aligned}Q_t^\pm&=P_x^\pm+[P^0,Q^\pm]\mp\lambda aP^\pm\\P_x^0&=[Q^+,P^-]+[Q^-,P^+].\end{aligned} \tag{2.4}$$

The polynomial (isospectral) flows can be found recursively by putting $P^{(N)}=\Sigma_0^N P_i\lambda^i$ to obtain a system of Nth-order evolution equations. The first two flows are

$$\pm aQ_{t_2}^\pm=Q_{xx}^\pm+[Q^\pm,[Q^\pm,Q^\mp]] \tag{2.5a}$$

$$a^2Q_{t_3}^\pm=Q_{xxx}^\pm+3[Q_x^\pm,[Q^\pm,Q^\mp]]. \tag{2.5b}$$

In terms of the coordinates q^α and $r^{-\alpha}$ these equations are written (using the summation convention):

$$aq_{t_2}^\alpha=q_{xx}^\alpha+R^\alpha{}_{\beta\gamma-\delta}q^\beta q^\gamma r^{-\delta} \tag{2.6a}$$

$$a^2q_{t_3}^\alpha=q_{xxx}^\alpha+3R^\alpha{}_{\beta\gamma-\delta}q_x^\beta q^\gamma r^{-\delta} \tag{2.6b}$$

with corresponding equations for $r^{-\alpha}$ and where $R^{\alpha}{}_{\beta\gamma-\delta}$ are the components of the curvature tensor of the corresponding symmetric space, given by $[e_\beta,[e_\gamma,e_{-\delta}]]=R^{\alpha}{}_{\beta\gamma-\delta}e_\alpha$. The Hermitian symmetric spaces are listed in Helgason (1978), thus providing a classification of all such equations. There are four infinite families, labelled by the Cartan classification: AIII, CI, DIII, BDI. The generalised NLS equations (t_2 flow, usually written in the real form $r^{-\alpha}=\pm(q^\alpha)^*$) were considered in detail in Fordy and Kulish (1983). In this paper we are mainly interested in the t_3 flow. The inverse scattering transform for the linear problem (2.1), and hence the solution of all the above equations, has been discussed by Gerdjikov (1986).

3. Reductions: generalised KdV and MKdV equations

In this section we concentrate on the third-order flow (2.5*b*) which, in all but class BDI symmetric spaces, can be written

$$a^2Q^+_{t_3}=Q^+_{xxx}-3(Q^+_xQ^-Q^++Q^+Q^-Q^+_x) \tag{3.1}$$

with a similar equation for Q^-. We consider the two reductions: Q^- constant, corresponding to matrix KdV equations, and $Q^-=\pm(Q^+)^{\mathrm{T}}$ (superscript T meaning matrix transpose), corresponding to generalised MKdV equations. The first case corresponds to $r^{-\alpha}$ being constant so that, by use of the Bianchi identities, (2.6*b*) can be written

$$a^2q^\alpha_{t_3}=(q^\alpha_{xx}+\tfrac{3}{2}R^{\alpha}{}_{\beta\gamma-\delta}q^\beta q^\gamma r^{-\delta})_x \tag{3.2a}$$

while the second corresponds to $r^{-\alpha}=\pm q^\alpha$, so that (2.6*b*) takes the form

$$a^2q^\alpha_{t_3}=q^\alpha_{xxx}\pm3R^{\alpha}{}_{\beta\gamma-\delta}q^\beta_xq^\gamma q^\delta. \tag{3.2b}$$

3.1. Matrix KdV equations

When Q^- is a constant matrix, we can multiply equation (3.1) from the left by Q^- to obtain

$$a^2U_{t_3}=U_{xxx}-3(U^2)_x \tag{3.3}$$

for the matrix $U=Q^-Q^+$. A similar manipulation with the t_5 equation for Q^+ leads to the fifth-order equation:

$$a^4U_{t_5}=[U_{xxxx}-5(UU_{xx}+U_{xx}U)-5U_x^2+10U^3]_x. \tag{3.4}$$

These are just two of the isospectral flows of a matrix Schrödinger equation such as (1.1). Indeed, it is possible to derive this Schrödinger equation from the spectral problem (2.1). In all but the class BDI symmetric spaces the spectral problem (2.1) has block diagonal form:

$$\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}_x=\left[\begin{array}{c|c}a\lambda I_m & \mathbf{q}\\ \hline \mathbf{r} & b\lambda I_n\end{array}\right]\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \tag{3.5}$$

where $\mathbf{q}$ and $\mathbf{r}$ are $m\times n$ and $n\times m$ matrices, respectively, I_m is the $m\times m$ identity matrix and ψ_1 and ψ_2 are m- and n-dimensional column vectors. Since $\mathrm{Tr}A=0$ we have $ma+nb=0$. The matrix

$$U=\left(\begin{array}{c|c}0 & 0\\ \hline 0 & \mathbf{rq}\end{array}\right)$$

so that $\mathbf{u}=\mathbf{rq}$ is a square $(n\times n)$ matrix which also satisfies equations (3.3) and (3.4). A simple manipulation of (3.5) leads to

$$\boldsymbol{\psi}_{2xx}-(a+b)\lambda\boldsymbol{\psi}_{2x}-\mathbf{u}\boldsymbol{\psi}_2=-ab\lambda^2\boldsymbol{\psi}_2 \tag{3.6}$$

which, by means of $\boldsymbol{\psi}_2=\exp[-\frac{1}{2}(a+b)\lambda x]\boldsymbol{\phi}$, is transformed into

$$\boldsymbol{\phi}_{xx}-\mathbf{u}\boldsymbol{\phi}=\tfrac{1}{4}(a-b)^2\lambda^2\boldsymbol{\phi}. \tag{3.7}$$

When $m=n$, $a+b=0$ so that this is the identity transformation and (3.6) and (3.7) are identical. This condition can hold in class AIII and always holds in classes CI (in which case $\mathbf{q}$ and $\mathbf{r}$ are square and symmetric) and DIII (in which case $\mathbf{q}$ and $\mathbf{r}$ are square and skew-symmetric). In the class AIII it is usual that $m\neq n$, so that without loss of generality $m<n$, in which case (3.3) consists of n^2 scalar equations whereas (2.6*b*) represents $mn(<n^2)$ equations. Indeed, in both cases there are only mn independent functions q^α, making the system (3.3) degenerate. This is a reduction of the n^2 independent equations derived from (3.5) when $m=n$.

Furthermore, the same manipulation can be carried out for $W=Q^+Q^-$ (or equivalently the $m\times m$ matrix $\mathbf{w}=\mathbf{qr}$). Here we have only $m^2<mn$ independent equations, this time for certain linear combinations of the mn independent variables.

3.2. *Generalised* MKdV *equations*

We now consider the reduction $Q^-=(Q^+)^{\mathrm{T}}$, which corresponds to $r^{-\alpha}=q^\alpha$ and to equation (3.2*b*). The RHS of this equation is not an exact derivative, so (3.2*b*) is not in conservation form. In all but class BDI the spectral problem has block diagonal form:

$$\begin{bmatrix}\boldsymbol{\psi}_1\\ \boldsymbol{\psi}_2\end{bmatrix}_x=\left[\begin{array}{c|c}a\lambda I_m & \mathbf{v}\\ \hline \mathbf{v}^{\mathrm{T}} & b\lambda I_n\end{array}\right]\begin{bmatrix}\boldsymbol{\psi}_1\\ \boldsymbol{\psi}_2\end{bmatrix} \tag{3.8}$$

where $\mathbf{v}$ is an $m\times n$ matrix, I_m is the $m\times m$ identity matrix and $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are m- and n-dimensional column vectors. In terms of $\mathbf{v}$, equation (3.1) is

$$a^2\mathbf{v}_{t_3}=\mathbf{v}_{xxx}-3(\mathbf{v}_x\mathbf{v}^{\mathrm{T}}\mathbf{v}+\mathbf{v}\mathbf{v}^{\mathrm{T}}\mathbf{v}_x). \tag{3.9}$$

3.3. *Generalised Miura transformations*

In the case when $\mathbf{v}$ is a square $(n\times n)$ and symmetric matrix we can change the basis so that (3.8) takes the form

$$\begin{bmatrix}\boldsymbol{\phi}_1\\ \boldsymbol{\phi}_2\end{bmatrix}_x=\left[\begin{array}{c|c}\mathbf{v} & \lambda I_n\\ \hline \lambda I_n & -\mathbf{v}\end{array}\right]\begin{bmatrix}\boldsymbol{\phi}_1\\ \boldsymbol{\phi}_2\end{bmatrix} \tag{3.10}$$

which implies that

$$(\partial+\mathbf{v})(\partial-\mathbf{v})\boldsymbol{\phi}_1=\lambda^2\boldsymbol{\phi}_1. \tag{3.11}$$

If we define $\mathbf{u}$ by

$$\mathbf{u}=\mathbf{v}_x+\mathbf{v}^2 \tag{3.12}$$

then (3.11) takes the form of the matrix Schrödinger equation (3.7). This is a generalised Miura transformation (Fordy and Gibbons 1980, 1981) which maps isospectral flows of (3.10) into those of (3.7). In particular the generalised MKdV equation (3.9), with $\mathbf{v}^{\mathrm{T}}=\mathbf{v}$ is mapped onto the generalised KdV equation (3.3). Since $\mathbf{u}$ must also be symmetric in this case, both these spectral problems are related to class CI symmetric spaces.

4. Conserved densities and Hamiltonian structure

For the general system of § 2, the Hamiltonian structure was given in Fordy and Kulish (1983):

$$q^{\alpha}_{t_n}=g^{\alpha-\beta}\frac{\delta H_n}{\delta r^{-\beta}} \qquad r^{-\alpha}_{t_n}=-g^{-\alpha\beta}\frac{\delta H_n}{\delta q^{\beta}} \tag{4.1}$$

with $H_n=\int \mathcal{H}_n\,\mathrm{d}x$, the conserved densities $\mathcal{H}_n$ being defined below. From the equations of motion (2.6) and (4.1) we can calculate the first few densities 'by hand' in terms of the metric and curvature tensors of the symmetric space:

$$\mathcal{H}_0=\sum_{\alpha,\beta} g_{\alpha-\beta}q^{\alpha}r^{-\beta} \tag{4.2a}$$

$$\mathcal{H}_1=\tfrac{1}{2}\sum_{\alpha,\beta} g_{\alpha-\beta}(q^{\alpha}_x r^{-\beta}-q^{\alpha}r^{-\beta}_x) \tag{4.2b}$$

$$a\mathcal{H}_2=-\sum_{\alpha,\beta} g_{\alpha-\beta}q^{\alpha}_x r^{-\beta}_x+\tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} R_{-\alpha\beta\gamma-\delta}r^{-\alpha}q^{\beta}q^{\gamma}r^{-\delta} \tag{4.2c}$$

$$a^2\mathcal{H}_3=\tfrac{1}{2}\sum_{\alpha,\beta} g_{\alpha-\beta}(q^{\alpha}_{xxx}r^{-\beta}-q^{\alpha}r^{-\beta}_{xxx})-\tfrac{3}{4}\sum_{\alpha,\beta,\gamma,\delta} R_{-\alpha\beta\gamma-\delta}(r^{-\alpha}_x q^{\beta}q^{\gamma}r^{-\delta}-r^{-\alpha}q^{\beta}_x q^{\gamma}r^{-\delta}) \tag{4.2d}$$

$$a^3\mathcal{H}_4=\sum_{\alpha,\beta} g_{\alpha-\beta}q^{\alpha}_{xx}r^{-\beta}_{xx}-3\sum_{\alpha,\beta,\gamma,\delta} R_{-\alpha\beta\gamma-\delta}r^{-\alpha}_x q^{\beta}_x q^{\gamma}r^{-\delta}-\tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} R_{-\alpha\beta\gamma-\delta}(r^{-\alpha}q^{\beta})_x(q^{\gamma}r^{-\delta})_x$$
$$+\tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta,\varepsilon,\rho,\sigma} R_{-\alpha\beta\gamma-\delta}R^{\gamma}{}_{\varepsilon\rho-\sigma}r^{-\alpha}q^{\beta}r^{-\delta}q^{\varepsilon}q^{\rho}r^{-\sigma}. \tag{4.2e}$$

These densities can be generated systematically (Athorne and Fordy 1986) in terms of the asymptotic expansion $P=\Sigma_{i=0}^{\infty}P_i\lambda^{-i}$ satisfying (2.3):

$$a^{n-1}\mathcal{H}_n=\frac{1}{n+1}\operatorname{Tr}AP_{n+2}. \tag{4.3}$$

However, in this paper we are mainly interested in the reduced systems of § 3.

4.1. Generalised MKdV *equations*

Both reductions of § 3 invalidate the Hamiltonian structure (4.1), even though the conserved densities (4.2), in reduced form, still exist. For the reduction $r^{-\alpha}=q^{\alpha}$, the odd numbered conserved densities vanish identically whilst the even densities are generalisations of the familiar MKdV ones. However, despite the infinite number of conserved densities, the hierarchy to which (3.9) belongs is not Hamiltonian WRT the simpler possible Hamiltonian structures.

4.2. Matrix KdV equations

We start with the matrix Schrödinger equation

$$(\partial^2 - U)\psi = \tfrac{1}{4}\lambda^2\psi \tag{4.4}$$

where U is a $N\times N$ matrix of functions and ψ is an N-dimensional column vector. Let the jth-order differential operator T_j correspond to the jth time, t_j, so that

$$\psi_{t_j} = T_j\psi. \tag{4.5}$$

Corresponding to the matrix KdV equation (1.2) we have

$$T_3 = 4\partial^3 - 6U\partial - 3U_x. \tag{4.6}$$

Let Ψ be the fundamental matrix of solutions of (4.4) and define matrix R by

$$R = \Psi_x\Psi^{-1} \tag{4.7}$$

so that

$$R_x + R^2 = U + \tfrac{1}{4}\lambda^2 I_N \tag{4.8}$$

and

$$R_{t_n} = (\Psi_{t_n}\Psi^{-1})_x + (\Psi_{t_n}\Psi^{-1})R - R\Psi_{t_n}\Psi^{-1} \tag{4.9}$$

so that

$$\frac{\partial}{\partial t_n}\operatorname{Tr} R = \frac{\partial}{\partial x}\operatorname{Tr}(\Psi_{t_n}\Psi^{-1}). \tag{4.10}$$

Thus $\int(\operatorname{Tr} R)\,\mathrm{d}x$ is constant with respect to each of the t_n.

In the usual way we consider the asymptotic expansion $R = \Sigma_{n=-1}^{\infty} R_n\lambda^{-n}$. The Riccati equation (4.8) leads to

$$\begin{aligned} &R_{-1} = \tfrac{1}{2}I_N \qquad R_0 = 0 \qquad R_1 = U \\ &R_{n+1} = -R_{nx} - \sum_{i=1}^{n} R_i R_{n-i} \end{aligned} \tag{4.11}$$

and (4.10) implies that each of the $\operatorname{Tr} R_n$ is conserved. Each of the even terms, R_{2n}, is an exact derivative, so leads to a trivial conserved density. The Hamiltonians are numbered so that H_{2n+1} is proportional to $\int R_{2n+3}\,\mathrm{d}x$. After simplification the first few are

$$\begin{aligned} &H_{-1} = \int \operatorname{Tr} U\,\mathrm{d}x \qquad aH_1 = \int \operatorname{Tr}(\tfrac{1}{2}U^2)\,\mathrm{d}x \\ &a^2H_3 = \int \operatorname{Tr}(-\tfrac{1}{2}U_x^2 - U^3)\,\mathrm{d}x. \end{aligned} \tag{4.12}$$

$\mathscr{H}_{2n+1}$ is normalised so that the quadratic term is $\mathrm{Tr}(\frac{1}{2}(-1)^n U_{nx}^2)$. The equations of motion corresponding to H_{2n+1} are

$$U_{ijt_{2n+1}} = \partial \frac{\delta}{\delta U_{ji}} H_{2n+1}. \tag{4.13}$$

The matrix KdV (3.3) corresponds to H_3.

Remark. In the reduction $r^{-\alpha}$ = constant, the odd densities of (4.2) are exact derivatives whilst the even ones correspond to (4.12). In particular, under this reduction $\mathscr{H}_4$ corresponds to H_3 of (4.12) (the numbering is different because of the ∂ in (4.13)).

In § 3 we derived the matrix Schrödinger equation (3.7) from the block diagonal spectral problem (3.5), with the potential $\mathbf{u}$ being the $n \times n$ matrix $\mathbf{rq}$. It was pointed out that a similar manipulation leads to a matrix Schrödinger equation with the potential $\mathbf{u}$ being the $m \times m$ matrix $\mathbf{qr}$. If $m \leq n$, then both these matrices have rank m.

The coefficients R_k, given by (4.11), are polynomial in $\mathbf{u}$ and its x derivatives. Depending upon which of the two Schrödinger operators is in use, let us write $R_k[\mathbf{rq}]$ or $R_k[\mathbf{qr}]$. Then, since $\boldsymbol{r}$ is constant, it is easily seen that $\mathrm{Tr}\, R_k[\mathbf{rq}] \equiv \mathrm{Tr}\, R_k[\mathbf{qr}]$, so that the two Schrödinger operators associated with the one spectral problem (3.5) have *identical* conserved densities. In each case the conserved density is a polynomial function of the matrix components U_{ij} and their x derivatives. Thus, for each conserved density we have polynomial expressions in two (usually) distinct sets of coordinates: $(\mathbf{rq})_{ij}$ and $(\mathbf{qr})_{ij}$. The consequences of this are discussed in the following examples.

AIII. In this case the spectral problem is of the form (3.5), with $\mathbf{q}$ an arbitrary $m \times n$ matrix. The compact real form is $\mathrm{SU}(m+n)/[\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))]$ and corresponds to the condition $r^{-\alpha} = -(q^\alpha)^*$, which is suitable for the discussion of NLS type equations (Fordy and Kulish 1983).

Let $\boldsymbol{\rho} = \mathbf{qr}$ and $\boldsymbol{\sigma} = \mathbf{rq}$. Then the Hamiltonian H_3 of (4.12) can be written in terms of the coordinates σ_{ij} and ρ_{ij} (using the summation conversion) with

$$a^2 \mathscr{H}_3 = -\tfrac{1}{2}\sigma_{ijx}\sigma_{jix} - \sigma_{ij}\sigma_{jk}\sigma_{ki} \equiv -\tfrac{1}{2}\rho_{ijx}\rho_{jix} - \rho_{ij}\rho_{jk}\rho_{ki}. \tag{4.14}$$

Equations (4.13) are thus

$$a^2 \sigma_{ijt_3} = (\sigma_{ijxx} - 3\sigma_{ik}\sigma_{kj})_x \tag{4.15}$$

and similarly for ρ_{ij}. Since there are (generally) only mn individual functions in $\mathbf{q}$ the system of equations for σ_{ij} is apparently overdetermined. However, σ has only m independent row vectors so that there are precisely mn independent equations (4.15), which can always be rearranged as equations (3.2a) for q^α.

Remark. If rank $\mathbf{q} = m$ but rank $\mathbf{r} < m$, then rank $\boldsymbol{\sigma} < m$ and the system (4.15) is underdetermined for q^α.

The corresponding m^2 equations for ρ_{ij} are always underdetermined for q^α unless $m = n$.

The apparently degenerate (when $m < n$) Hamiltonian system (4.15) for $\boldsymbol{\sigma}$ is a consistent reduction of the corresponding system for the case $m = n$. Thus, when $m < n$ the system (3.2a) is squeezed between two Hamiltonian systems: one for $\boldsymbol{\sigma}$, too large but reduced, the other for $\boldsymbol{\rho}$, too small, being equations for certain linear combinations of the q^α. However, when $m < n$ the equations for q^α are not themselves Hamiltonian WRT a simple Hamiltonian structure, even though they possess an infinite number of conserved densities and an infinite number of commuting flows. When $m = n$ the system (3.2*a*) is Hamiltonian when $\mathbf{r}$ is chosen to be symmetric. In particular when $\mathbf{r}$ is the identity matrix, $\boldsymbol{\sigma} = \boldsymbol{\rho} = \mathbf{q}$.

Example. $1 = m < n$: vector KdV. Here $\sigma_{ij} = r_i q_j$ and without loss of generality $r_1 \neq 0$. Equation (4.15) for σ_{ij} now takes the form (after dividing by r_1):

$$a^2 q_{jt_3} = [q_{jxx} - 3(\boldsymbol{q}\cdot\boldsymbol{r})q_j]_x. \tag{4.16}$$

Since $\rho = \boldsymbol{q}\cdot\boldsymbol{r}$ is a scalar it satisfies the scalar KdV equations, which is also easily derived from (4.16).

Example. $2 = m \leqslant n$. Here

$$q = \begin{pmatrix} q_1 & \cdots & q_n \\ p_1 & \cdots & p_n \end{pmatrix} \qquad \text{and} \qquad r = \begin{pmatrix} r_1 & \cdots & r_n \\ s_1 & \cdots & s_n \end{pmatrix}^{\mathrm{T}}$$

so that $\sigma_{ij} = r_i q_j + s_i p_j$ and

$$\boldsymbol{\rho} = \begin{pmatrix} \boldsymbol{q}\cdot\boldsymbol{r} & \boldsymbol{q}\cdot\boldsymbol{s} \\ \boldsymbol{p}\cdot\boldsymbol{r} & \boldsymbol{p}\cdot\boldsymbol{s} \end{pmatrix}.$$

Since both matrices are of rank 2 we need only consider equations (4.15) for σ_{1j} and σ_{2j}. Multiplying these equations by $\left(\begin{smallmatrix} r_1 & s_1 \\ r_2 & s_2 \end{smallmatrix}\right)^{-1}$ leads to

$$\begin{aligned} a^2 q_{jt_3} &= q_{jxxx} - 3[(\boldsymbol{r}\cdot\boldsymbol{q})q_j + (\boldsymbol{s}\cdot\boldsymbol{q})p_j]_x \\ a^2 p_{jt_3} &= p_{jxxx} - 3[(\boldsymbol{r}\cdot\boldsymbol{p})q_j + (\boldsymbol{s}\cdot\boldsymbol{p})p_j]_x. \end{aligned} \tag{4.17}$$

The four equations for ρ_{ij} can be obtained by taking scalar products of the vector equations (4.17) with $\boldsymbol{r}$ and $\boldsymbol{s}$.

CI and DIII. When $m = n$ in class AIII it is possible to reduce the system (4.15) by taking $\mathbf{r}$ and $\mathbf{q}$ to be both symmetric or skew-symmetric. These correspond respectively to class CI and class DIII symmetric spaces.

Example. $m = n = 2$, symmetric. If in (4.17) we take $r_1 = s_2 = 1$, $r_2 = s_1 = 0$ and $p_1 = q_2$ then we have three independent equations (relabelling p_2 as q_3):

$$\begin{aligned} a^2 q_{1t_3} &= q_{1xxx} - 3(q_1^2 + q_2^2)_x \\ a^2 q_{2t_3} &= q_{2xxx} - 3(q_1 q_2 + q_2 q_3)_x \\ a^2 q_{3t_3} &= q_{3xxx} - 3(q_2^2 + q_3^2)_x. \end{aligned} \tag{4.18}$$

As a three-component system the Hamiltonian structure (4.13) is replaced by

$$a^2 \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}_{t_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \partial \begin{pmatrix} \delta H/\delta q_1 \\ \delta H/\delta q_2 \\ \delta H/\delta q_3 \end{pmatrix} \tag{4.19}$$

with $\mathscr{H}=-\frac{1}{2}(q_{1x}^2+q_{3x}^2)-q_{2x}^2-(q_1+q_3)^3+3(q_1+q_3)(q_1q_3-q_2^2)$. The matrix diag$(1,\frac{1}{2},1)$ is $g^{\alpha-\beta}$, the inverse of the metric tensor of the symmetric space.

Example $m=n=4$, skew symmetric. This reduction corresponds to the symmetric space inclusion

$$\frac{\mathrm{SO}(2n)}{\mathrm{U}(n)}\subset\frac{\mathrm{SU}(2n)}{\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n))}.$$

We label the components of $\mathbf{q}$ as

$$\begin{pmatrix} 0 & q_1 & q_3 & q_6 \\ -q_1 & 0 & q_2 & q_5 \\ -q_3 & -q_2 & 0 & q_4 \\ -q_6 & -q_5 & -q_4 & 0 \end{pmatrix}$$

and $\mathbf{r}$ similarly (but with the minus signs in the upper triangle). If we define vectors $\boldsymbol{q}_1=(q_1, q_2, q_3)$ and $\boldsymbol{q}_2=(q_4, q_6, -q_5)$ and similarly for $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, then equations (3.2a) (with $a=1$) take the form

$$\begin{aligned} \boldsymbol{q}_{1t_3}&=[\boldsymbol{q}_{1xx}+3(\boldsymbol{r}_1\cdot\boldsymbol{q}_1+\boldsymbol{r}_2\boldsymbol{q}_2)\boldsymbol{q}_1-3(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)\boldsymbol{r}_1]_x \\ \boldsymbol{q}_{2t_3}&=[\boldsymbol{q}_{2xx}+3(\boldsymbol{r}_1\cdot\boldsymbol{q}_1+\boldsymbol{r}_2\cdot\boldsymbol{q}_2)\boldsymbol{q}_2-3(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)\boldsymbol{r}_2]_x \end{aligned} \tag{4.20}$$

where the 'dot' refers to the usual scalar product. Two of the Hamiltonians (4.12) are

$$\begin{aligned} H_1&=\int[\tfrac{1}{4}(\mathrm{Tr}\,\mathbf{rq})^2+2(\det\mathbf{rq})^{1/2}]\,\mathrm{d}x \\ H_3&=\int[-\tfrac{1}{4}(\mathrm{Tr}\,\mathbf{rq}_x)^2+2(\det\mathbf{rq}_x)^{1/2}-\tfrac{1}{4}(\mathrm{Tr}\,\mathbf{rq})^3-3(\mathrm{Tr}\,\mathbf{rq})(\det\mathbf{rq})^{1/2}]\,\mathrm{d}x. \end{aligned} \tag{4.21}$$

There is no need for alarm at the appearance of square roots here. For a skew-symmetric matrix the determinant is a perfect square, the square root being the Pfaffian. Thus

$$\begin{aligned} (\det\mathbf{rq})^{1/2}&=(r_1r_4-r_3r_5+r_2r_6)(q_1q_4-q_3q_5+q_2q_6) \\ &\equiv(\boldsymbol{r}_1\cdot\boldsymbol{r}_2)(\boldsymbol{q}_1\cdot\boldsymbol{q}_2) \end{aligned}$$

so that the Hamiltonian is still polynomial in our variables. We do not know whether these Hamiltonians take such simple form for $n>4$. Note, however, that when n is odd the Pfaffiians of $\mathbf{q}$ and $\mathbf{r}$ are both zero.

BDI.

$$\frac{\mathrm{SO}(m+n)}{\mathrm{SO}(m)\times\mathrm{SO}(n)}\qquad m=2.$$

The linear spectral problem below is not of the block diagonal form (3.5), so that the given derivation of (3.1), (3.3) and (3.7) is invalid, although equation (3.2) still holds. The Hamiltonian structure (4.1) can still be used with Hamiltonians (4.2). The reductions of (4.2) obtained by setting $r^{-\alpha}$ constant are still valid, so that expressions of the form (4.12) still hold. However, there is no $U=Q^-Q^+$ which can be used in conjunction with (4.12) and (4.13). We present the case $n = 5$, so that the linear spectral problem takes the form:

$$\begin{pmatrix} \psi_1 \\ \cdot \\ \cdot \\ \cdot \\ \psi_7 \end{pmatrix}_x = \left(\begin{array}{c|ccc|ccc} 0 & r_1 & 0 & 0 & -q_1 & 0 & 0 \\ \hline q_1 & \lambda & q_2 & q_3 & 0 & q_4 & q_5 \\ 0 & r_2 & 0 & 0 & -q_4 & 0 & 0 \\ 0 & r_3 & 0 & 0 & -q_5 & 0 & 0 \\ \hline -r_1 & 0 & -r_4 & -r_5 & -\lambda & -r_2 & -r_3 \\ 0 & r_4 & 0 & 0 & -q_2 & 0 & 0 \\ 0 & r_5 & 0 & 0 & -q_3 & 0 & 0 \end{array}\right) \begin{pmatrix} \psi_1 \\ \cdot \\ \cdot \\ \cdot \\ \psi_7 \end{pmatrix}. \tag{4.22}$$

The first three equations of motion (3.2a) are

$$\begin{aligned} q_{1t_3} &= [q_{1xx} + 3(\boldsymbol{r}\cdot\boldsymbol{q})q_1 - 3f(\boldsymbol{q})r_1]_x \\ q_{2t_3} &= [q_{2xx} + 3(\boldsymbol{r}\cdot\boldsymbol{q})q_2 - 3f(\boldsymbol{q})r_4]_x \\ q_{3t_3} &= [q_{3xx} + 3(\boldsymbol{r}\cdot\boldsymbol{q})q_3 - 3f(\boldsymbol{q})r_5]_x \end{aligned} \tag{4.23}$$

where $\boldsymbol{r}\cdot\boldsymbol{q} = \Sigma_1^5 r_i q_i$ and $f(\boldsymbol{q}) = \frac{1}{2}q_1^2 + q_1 q_4 + q_3 q_5$. The remaining two equations are obtained by interchanging $2 \leftrightarrow 4$ and $3 \leftrightarrow 5$. The first two conserved quantities take the form

$$\begin{aligned} H_1 &= \int \tfrac{1}{2}(\boldsymbol{r}\cdot\boldsymbol{q})^2\,\mathrm{d}x \\ H_3 &= \int [-\tfrac{1}{2}(\boldsymbol{r}\cdot\boldsymbol{q}_x)^2 - (\boldsymbol{r}\cdot\boldsymbol{q})^3]\,\mathrm{d}x. \end{aligned} \tag{4.24}$$

Setting $q_1 = r_1 = 0$ gives the reduction: BDI with $n = 4$.

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